Abstract

We develop $L_1$ -norm model (That is always feasible and bounded) for ranking extreme efficient decision making units (DMUs) in stochastic data envelopment analysis (DEA), and also, we present a deterministic equivalent of stochastic model. It is shown that this deterministic model can be convert to a quadratic program.

Keywords: Ranking, Stochastic DEA, $L_1$ -norm

1. Introduction

In order to incorporate stochastic input and output variations into the DEA analysis Sengupta (1992)[5]For example, generalized the CCR ratio model by defining measure of the relative efficiency of a DMU as the maximum of the sum of the expected ratio of weighted outputs to weighted inputs and a reliability function subject to several chance constraints. Cooper et al (1996),[6], incorporated the satisfying concepts of Simon into DEA and developed the satisfying DEA model. More recently, stochastic input and output variations into DEA have been studied by: Asgharian, Khodabakhshi, and Neralic [7],Khodabakhshi [8], and Khodabakhshi and Asgharian[4], and Vesal et al [9].

Ranking of efficient DMUs is very important question and many DEA researchers and practitioners have studied about it. Anderson and Peterson [1] and Karimi Takalo [10] were first addressed this question in their seminal paper where they introduced super-efficiency models to rank efficient decision making units. Also Jahanshahlo et al.[2] introduced $L_1$ -norm model for ranking of efficient DMUs.

In this paper we extend Stochastic$L_1$ -norm model, allowing deterministic inputs and outputs to be stochastic. Then, We obtain a deterministic equivalent to our stochastic model and show this deterministic equivalent can be transformed to a quadratic programming model.

2. Background

We consider $n$ homogeneous DMUs $\{DMU_j|j=1,\ldots,n\}$ each having $m$ inputs denoted by $x_j \in \mathbb{R}^m$ ($j=1,\ldots,n$) and $s$ outputs denoted by $y_j \in \mathbb{R}^s$ ($j=1,\ldots,n$). We assume that $x_j$ and $y_j$ are non-negative deterministic elements. The production possibility set (PPS) is defined as follows:

$$T_c = \left\{ (x,y) \left| \sum_{j=1}^n \lambda_j x_j \leq x, \sum_{j=1}^n \lambda_j y_j \geq y, \lambda_j \geq 0, j = 1, \ldots, n \right. \right\}$$

Assume that $DMU_0$ is one of the extreme efficient DMUs. By omitting $DMU_0$ from $T_c$, we define the production possibility set $T'_c$ as:

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**Ranking extreme efficient DMUs in stochastic DEA**

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Journal of productivity and development 1(2) 2015:63-66
www.pdjourn.com
\[T'_c = \left\{ (x, y) \left| \sum_{j=1}^{n} \lambda_j x_j \leq x, \sum_{j=1}^{n} \lambda_j y_j \geq y, \lambda_j \geq 0, j = 1, \ldots, n, j \neq 0 \right. \right\} \]

The \( L_1 \) - norm model is one of the important models for ranking of efficient DMUs. This model was introduced by Jahanshahloo et al[1]. It is used to rank DMU \( j \) as follows:

\[
\text{Min } \mu_j(x, y) = \sum_{i=1}^{m} x_i - x_{i0} + \sum_{r=1}^{s} y_r - y_{r0} \quad (1)
\]

\[
\text{S.t. } \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i \quad n_j = 1 \quad j \neq 0 \quad \lambda_j \geq 0, j = 1, \ldots, n, j \neq 0
\]

It was proof that model (1) was converted to a linear program[2] as:

\[
\text{Min } \mu_j^0(x, y) = \sum_{i=1}^{m} x_i - x_{i0} + \sum_{r=1}^{s} y_r - y_{r0} + A \quad (2)
\]

\[
\text{S.t. } \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i \quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_r
\]

Where \( A = -\sum_{i=1}^{m} x_{i0} + \sum_{r=1}^{s} y_{r0} \)

Theorem 1: Model (2) is always feasible and bounded.
Proof: Refer to Jahanshahloo et al[2].

3. Stochastic \( L_1 \) - norm

We are going to develop the \( L_1 \) - norm model[2] in stochastic data envelopment analysis to rank extreme efficient DMUs. Following Cooper et al[3] and Shamsaddini[11] let \( \hat{x}_j = (\hat{x}_{1j}, \ldots, \hat{x}_{mj}) \) and \( \hat{y}_j = (\hat{y}_{1j}, \ldots, \hat{y}_{sj}) \) be random input and output related to DMU \( j = 1, \ldots, n \). Let also \( \bar{x}_j = (\bar{x}_{1j}, \ldots, \bar{x}_{mj}) \) and \( \bar{y}_j = (\bar{y}_{1j}, \ldots, \bar{y}_{sj}) \) show the corresponding vectors of expected values of inputs and outputs for DMU \( j \). Suppose that all input and output components are jointly normally distributed in the following chance constrained version of the stochastic model (2) with inequality constraints:

\[
\text{Min } R_c^0(x, y) = \mathbb{E}(\sum_{i=1}^{m} x_i - \sum_{r=1}^{s} y_{r} + A) \quad (3)
\]

\[
\text{S.t. } P(\sum_{j=1}^{n} \lambda_j \hat{x}_{ij} \leq \bar{x}_i) \geq 1 - \alpha
\]

\[
P(\sum_{j=1}^{n} \lambda_j \hat{y}_{rj} \geq \bar{y}_r) \geq 1 - \alpha
\]
Where $\alpha$ is a predetermined value between 0 and 1, represents the probability measure.

The corresponding stochastic version of model (3), including slack variables is as follows:

$$\text{Min } R_c^0(X, Y) = E\left(\sum_{i=1}^{m} \tilde{x}_i\right) - E\left(\sum_{r=1}^{s} \tilde{y}_r\right) + A$$

S.t. \begin{align*}
P\left(\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} + s_i^- \leq \tilde{x}_i\right) &= 1 - \alpha \\
P\left(\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - s_r^+ \geq \tilde{y}_r\right) &= 1 - \alpha
\end{align*}

### 4. Deterministic equivalent

In this section, we exploit the normality assumption to introduce deterministic equivalent to the model (4), that is:

$$\text{Min } R_c^0(X, Y) = E\left(\sum_{i=1}^{m} \tilde{x}_i\right) - E\left(\sum_{r=1}^{s} \tilde{y}_r\right) + A$$

S.t. \begin{align*}
\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} + s_i^- + \sigma_i^0(\lambda)\varphi^{-1}(\alpha) &= \tilde{x}_i \\
\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - s_r^+ + \sigma_i^1(\lambda)\varphi^{-1}(\alpha) &= \tilde{y}_r \\
\tilde{x}_i - t_i^+ + \text{var}(x_i)\varphi^{-1}(\alpha) &= x_{i0} \\
\tilde{y}_r + t_r^- + \text{var}(y_r)\varphi^{-1}(\alpha) &= y_{r0} \\
\tilde{x}_i &\geq 0, s_i^- \geq 0, t_i^+ \geq 0, i = 1, \ldots, m \\
\tilde{y}_r &\geq 0, s_r^+ \geq 0, t_r^- \geq 0, r = 1, \ldots, s \\
\lambda_j &\geq 0, j = 1, \ldots, n, j \neq 0
\end{align*}

Where $\varphi$ the cumulative distribution function of standard normal random variable and $\varphi^{-1}$ is its inverse, we assume that $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ are the means of the input and output variables, which can be estimated by the observed values of the inputs and outputs using the aforementioned property of normal distribution, one can show that

$$\left(\sigma_i^0(\lambda)\right)^2 = \sum_{j=1}^{n} \sum_{k=0}^{m} \lambda_k \text{Cov}(\tilde{x}_{ij}, \tilde{x}_{ik}) + 2(\lambda_0 - 1) \sum_{j=1}^{n} \lambda_j \text{Cov}(\tilde{x}_{ij}, \tilde{x}_{i0}) + (\lambda_0 - 1)^2 \text{Var}(\tilde{x}_{i0})$$

$$\left(\sigma_i^1(\lambda)\right)^2 = \sum_{k=0}^{m} \sum_{j=1}^{n} \lambda_k \lambda_j \text{Cov}(\tilde{y}_{rj}, \tilde{y}_{rk}) + 2(\lambda_0 - 1) \sum_{k=1}^{n} \lambda_k \text{Cov}(\tilde{y}_{rk}, \tilde{y}_{r0}) + (\lambda_0 - 1)^2 \text{Var}(\tilde{y}_{r0})$$

It is obvious, from the form of $\sigma_i^0(\lambda), \sigma_i^1(\lambda)$, that model is a quadratic program.

### 4. Conclusion

Stochastic models may be better suited for DEA when there is uncertainty associated with the inputs and/or outputs of DMUs or when an analyst may be wondering how much change can be incurred in the ranking of...
DMUs if sum of the inputs and/or outputs change. In this paper, we have developed Stochastic $L_1$-norm model for ranking extreme efficient DMUs in stochastic data envelopment analysis. We have obtained the deterministic equivalent for the stochastic version. Applying the proposed approach in different norms, practically, would be interesting for further research.

References